Fall 2012

MA635. General information

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Recommended textbook: Royden, Fitzpatrick “Real Analysis.”
Additional reading: Kolmogorov, Fomin “Elements of the Theory of Functions and Functional Analysis” (vol.2). (Avoid Silverman’s “translation.”)

In general, for up to date information about the course, check the webpage `http://personal.stevens.edu/~anikolae/teaching/`

Material covered in the course:

(1) Preliminaries
- Naive set theory: set-theoretic operations and basic identities, mappings, Axiom of Choice and Zorn’s Lemma (no proof), countable and uncountable sets, cardinality.
- Relations: partial and total orders, equivalence relations, equivalence classes, quotient set of an equivalence relation.

(2) Real Number system
- Axiomatization of real numbers: field axioms, order axioms, completeness axiom. Archimedian property. Extended real numbers.
- Open and closed sets in $\mathbb{R}$; σ-algebra of subsets; Borel sets in $\mathbb{R}$.
- Sequences in $\mathbb{R}$: limit of a sequence, limit superior and limit inferior, basic properties; Cauchy sequences, Cauchy criterion.
- Completeness axiom and its equivalent forms: Heine–Borel theorem, Nested set theorem, Monotone convergence theorem for $\mathbb{R}$, Bolzano–Weierstrass theorem.
- *(Optional)* Construction of real numbers via Dedekind cuts (after Rudin).
- Continuous functions $\mathbb{R} \to \mathbb{R}$: definitions ($\varepsilon$–$\delta$ and topological) and basic properties, the Extreme value theorem, the Intermediate value theorem, uniform continuity, uniform continuity of a function continuous on a closed bounded set, monotone functions, jump of a monotone function.

(3) Lebesgue measure and integral (mostly, after Caratheodory)
- Legesgue outer measure, basic properties: outer measure extends length, is translation invariant and countable subadditive.
- Lebesgue measurable sets: definition (Caratheodory). Measurability of outer measure 0 sets. σ-algebra of measurable sets.
- Inner (closed) and outer (open) approximation of Lebesgue measurable sets.
- Lebesgue measure on $\mathbb{R}$, basic properties: Lebesgue measure extends length, is tranlation invariant and countable additive. Continuity of the Lebesgue measure. Corollaries: finite additivity, monotonicity, excision property, countable monotonicity of the Lebesgue measure.
- Nonmeasurable sets: Vitali theorem. Cantor set, generalized Cantor set, their measure and basic properties. Cantor–Lebesgue function and existence of a non-Borel measurable set.

(4) Lebesgue measurable functions
- Lebesgue measurable functions: definition, equivalent conditions. Basic properties: measurability of linear combination, product, composition with a continuous function.
- Convergence of function sequences: pointwise convergence, pointwise a.e. convergence, uniform convergence. Measurability of pointwise a.e. limit. Characteristic function of a set, simple functions, simple approximation lemma and simple approximation theorem.
- Littlewoods principles. Egoroff's theorem. Lusin's theorem.

(5) Lebesgue integration
- Riemann integral via Darboux sums. Limitations of Riemann integral: Dirichlet function, Riemann integral of a pointwise limit of sequence of functions.
- Lebesgue integral of a bounded function over a set of finite measure: definition via upper and lower Lebesgue integral. Linearity and monotonicity, domain additivity. Connection between Riemann and Lebesgue integrals. Bounded convergence theorem for Lebesgue integral. Lebesgue integrability of a bounded function over a finite measure set, integrability of a bounded measurable function over a finite measure set.
- Lebesgue integral of a measurable nonnegative function. Linearity and monotonicity, domain additivity. Lebesgue integrable nonnegative measurable functions.
- The Lebesgue integral on $\mathbb{R}$ in general case. Lebesgue integrable measurable functions. Integral comparison test. Linearity and monotonicity, additivity over domain.
- Lebesgue integral and sequential convergence: Fatou's lemma. Monotone Convergence theorem for Lebesgue integral, the Lebesgue Dominated Convergence theorem. Corollaries: countable domain additivity and continuity of Lebesgue integral.

(6) Metric spaces and applications
- Metric spaces: definition of a metric and a metric space, basic examples, pseudometrics, equivalence of metrics.
- Normed spaces: definition of a norm and a normed space, basic examples, pseudonorms, equivalence of norms. Metric induced by a norm. $\|\cdot\|_{\max}$ norm and $C[a,b]$ as a normed space. $\|\cdot\|_p$ pseudonorm ($1 \le p < \infty$, no proof of triangle inequality) and $L^p(X)$, $X \subseteq \mathbb{R}$ as normed spaces.
- Basic topology in metric spaces: open set, closed sets, convergence of sequences. Continuous maps between metric spaces (ε–δ definition and topological definition), uniform continuity.
- Complete metric spaces: Cauchy sequences, definition of a complete metric space, examples. Completness of $C[a,b]$. Cantor intersection theorem.
- Compact metric spaces. Equivalence of Heine–Borel compactness, sequential compactness and completeness/total boundedness. Extreme value theorem.
- Banach contraction principle. Application: Picard local existence/uniqueness theorem.
- Baire category theorem. Dence, nowhere dense sets. Application: pointwise bounded family of continuous functions is uniformly bounded on an open set.

(7) Abstract measures (after Kolmogorov–Fomin, as presented in Dyachenko–Ulyanov)

Note. Proofs in this chapter were mostly cut due to bad weather issues.

- Notion a semiring of sets, ring of sets, σ-algebra of sets.
- Definition of a measure and premeasure. Abstract measure spaces. Basic examples.
- Finite premeasures on a semiring. Extension of a premeasure on a semiring to the ring generated by the semiring. σ-additivity carries under such extension.
- Outer measures: the Lebesgue outer measure and the Jordan outer measure. Lebesgue measurable sets: definition, basic properties. σ-algebra of measurable sets. Extension of a finite premeasure on semiring to the σ-algebra of Lebesgue measurable sets. σ-additivity carries under such extension.
- σ-finite premeasures. Extension of a σ-finite premeasure on semiring to the σ-algebra of Lebesgue measurable sets. σ-additivity carries under such extension. Example: rectangular regions in $\mathbb{R}^n$ and the Lebesgue measure on $\mathbb{R}^n$.